PROPERTIES OF EUCLIDEAN FIELDS
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ABSTRACT
Our main aim here is to discuss Euclidean fields and the properties of Euclidean fields. We discuss
that the fundamental theorem of arithmetic is rational theory may be generalized to algebraic
number theory. We discuss here that the expression of an integer as a product of primes is unique,
except from the order of the primes, the presence of units and ambiguities between associated
primes.

INTRODUCTION
A quadratic field \( k (\sqrt{m}) \) is called Euclidean if its ring of integers \( R \) has the Property that for any
elements \( \alpha, \beta \) of \( R \) with \( \beta \neq 0 \),
\[ \alpha = \beta \gamma + \delta, \quad |N(\delta)| < |N(\beta)| \]
for such fields there exists a there exists \( \gamma, \delta \) in \( R \) Euclid algorithm analogous to that for ordinary rational integers.

It was proved by Chatland and Devenport in 1950 are independently by Inkery at about
the same time that there and precisely 21 Euclidean fields \( K (\sqrt{m}) \), with norm function called
norm–Euclidean fields, viz
\[ m = -1, -2, -3, -7, 11, 15, 19, 23, 31, 43, 67, 119, 167, 229, 251, 353, 379, 401, 521, 821, 971 \]
are simple but there is no Euclidean algorithm in
these case. For Simplicity of a field, the condition of possessing a Euclidean algorithm is thus
sufficient but not necessary, that is a field can be simple without being norm Euclidean.

According to fundamental theorem in algebraic number theory
the expression of an integer
as a product of primes is unique. All the imaginary
quadratic fields \( K (\sqrt{m}) \) which have unique
factorization property are known.

They are given by
\[ m = -1, -2, -3, -7, 11, 15, 19, 23, 31, 43, 67, 119, 167, 229, 251, 353, 379, 401, 521, 821, 971 \]
and they are nine in numbers out of which first five are
Euclidean and the next four are non Euclidean.

The concept of greatest common divisor (G.C.D.) in the rational theory depends upon
division algorithm. According to division algorithm if a and b are arbitrary, integers a>0 there exist
q and r such that \( b = qa + r, 0 < r < a \)
Exactly as in rational theory we define G.C.D for algebraic integers as follow:
If \( \gamma \) is a common divisor of \( \gamma \) and \( \gamma_1 \) is a divisor of \( \gamma \), then \( \gamma \) is the greatest common
divisor of \( \gamma \) and \( \gamma_1 \) and is and write \( \gamma = (\gamma, \gamma_1) \)

Theorem
1. Any integer, not zero or a unit, is a product of primes.
For, if \( \gamma \) is not 0, or a unit it is divisible by a prime \( \pi_1 \)
Hence
\[ \gamma = \pi \gamma \quad [N(\gamma_1) < N(\gamma \pi)] \]
Either \( \gamma_1 \) is a unit or
\[ \gamma_1 = \gamma_1 \pi_2 \quad [N(\gamma_2) < N(\gamma_1)] \]
Counting in this way we get a decreasing sequence
\[ N(\gamma_1), N(\gamma_2), N(\gamma_3), \ldots \ldots \] of positive rational integers Hence \( N(\gamma) = 1 \) for some \( \gamma \) and \( \gamma_1 \)
is a unit C and hence
\[ \gamma = \pi_1 \pi_2 \ldots \ldots \ldots \pi c = \pi_1 \pi_2 \ldots \ldots \ldots \pi r \]
Where \( \pi \gamma = \pi \gamma c \) is an associate of \( \pi \gamma \) and so is itself a prime
Theorem²: R is a Unique factorization domain if and only of every irreducible elements of R is
also a prime in R
We know that if factorization in $R$ is unique and if $\pi$ is an irreducible element such that $\pi$ divides $\alpha\beta$ for some $\alpha, \beta \in R$, then $\pi$ must be associate of one of the irreducible factors of $\alpha$ or $\beta$ and so $\pi$ divides $\alpha$ or $\beta$ as required. Conversely, if every irreducible element is also a prime, then we can argue as in fundamental theorem of arithmetic in rational theory that if

$$\gamma = \pi_1 \pi_2 \ldots \pi_r$$

As a product of irreducible elements and if $\pi$ is an irreducible element occurring in another factorization of $\gamma$ for some $I$, where $\pi$ and $\pi_i$ are associates, and assuming that the result holds for $\gamma/\pi$ the required uniqueness factorization follows by induction.

Theorem 3: The fundamental theorem is true for any Euclidean field i.e a Euclidean field $k(\sqrt{m})$ has unique factorization

To prove this theorem, it is sufficient to show that every irreducible element $\pi$ of $R$ is a prime.

Let us suppose that $\pi$ divides $\alpha\beta$ but that $\pi$ does not divide $\alpha$. Then there exist $\rho$ and $\mu$ in $R$ such that

$$\alpha \rho + \pi \beta \mu = 1$$

Whence $\alpha\beta \rho + \pi \beta \mu = \beta$

Hence $\pi$ divides $\beta$ thus $\pi$ is prime and the theorem follows

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