PERIODIC SOLUTION OF THE RESTRICTED THREE BODY PROBLEM

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ABSTRACT
The effect of perturbation in Coriolis force on the periodic solution of the restricted three body problem representing analytic continuation of Keplerian rectilinear motion has been examined. However, we have taken the perturbation in the Coriolis force to be of the order of reduced mass. We had to calculate the 1st order perturbations.

INTRODUCTION:
In this paper we have studied the generalization of the restricted three body problem introducing perturbation in the Coriolis force. We examined the existence of the periodic solution taken the perturbation to be the order of the reduced mass of the similar primary. We have also calculated 1st order perturbation.

1. Hamiltonian Equation of Motion:
Using non-dimensional variable and synodic system of co-ordinates, the Hamiltonian equation of motion are
\[
\frac{dx}{dt} = \frac{\partial H}{\partial x}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial y} \\
\frac{dx}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial y}
\]

\[H = \frac{1}{2}(x^2 + y^2) + (yx - xy) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} (2)
\]
\[r_2^1 = (x - \mu + 1) + y^2 \quad (3)
\]

We consider the perturbation in the Coriolis force with the help of the parameter \(\alpha\), the unperturbed value of being unity. The corresponding Hamiltonian function takes the following form
\[H = 1/2(x^2 + y^2) + \alpha(yx - xy) + 1/2(\alpha^2 - 1)(x^2 + y^2) - \frac{1-\mu}{r_1} - \frac{\pi}{r_2} \quad (2)
\]
Here \(\alpha\) may be taken as
\[\alpha = 1 + \epsilon, \, \epsilon \ll 1
\]
Where \(\epsilon\) represent the perturbation in the Coriolis up to first order, we have
\[H = 1/2(x^2 + y^2) + (1+\epsilon)(yx - xy) + \epsilon (x^2 + y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \quad (4)
\]

2. Limiting case = 0

For the elimination of singularity at the bigger primary, we have.
The regularized canonical equation of motion becomes:
\[
\frac{dp}{ds} = \frac{\partial \Omega}{\partial p}, \quad \frac{dQ}{ds} = \frac{\partial \Omega}{\partial Q} \\
\frac{dp}{ds} = -\frac{\partial \Omega}{\partial p}, \quad \frac{dQ}{ds} = \frac{\partial \Omega}{\partial Q}
\]
Where
\[ \Omega = \frac{1}{2} (p^2 + Q^2) + 2r_1(qp - pQ) + 4\mu (p^2 - q^2) - 4 + 4\mu - 4\mu \frac{r_1}{r_2} + 2c r_1 + 2 \in r_1(qp - pQ) + 2\mu (pQ + qp) - 4\epsilon \mu (p^4 - q^4) + 4\epsilon r^3 (7) \]

\[ r_1 = p^2 + q^2, r_2^2 = 1 - 2(p^2 + q^2) + (p^2 + q^2) (8) \]

The solution of the system (6), being the solution of the system (1), we have
\[ \Omega = \Omega (9) \]

The sidereal system of co-ordinate corresponds to collinear motion has the form
\[ P_0 = \rho \sin Q q_0 = \rho \cos Q \]
\[ P_0 = \rho \sin Q Q_0 = \rho \cos Q \]
\[ (10) \]

The period of such solution is
\[ \delta = \frac{\pi k}{2\sqrt{c}} \text{ if } (K+M) \text{ is an even number} \]
\[ \frac{\pi k}{\sqrt{2}} \text{ if } (K+M) \text{ is an odd number} \]
\[ (12) \]

The relation between \( t \) and \( s \) for \( \mu = 0 \) is given by
\[ t_0 = 2(\theta - \omega) = \frac{1}{n} 4\sqrt{c} s + \sin(4\sqrt{c}s - 2\phi_0) \]
\[ (14) \]

3. Proof of Isoperiodic solution:
We see that Hamiltonian function (7) possesses the same properties. Therefore, under the conditions
\[ q(0) = 0, p(s^*) = Q, (S^*) = 0, s^* = \frac{s}{4} \]
\[ (15) \]
and \( q(0) = 0, q(s^*) = p(s^*) = 0, s^* = \frac{s}{2} \]
\[ (16) \]
The general solution \( x_i(s^*) \), \( i = 1, 2, 3, 4 \) are periodic with periods. where \( x_i (i = 1, 2, 3, 4) \), \( p, q, P, Q \) are respectively denoted
Form (15) where \( k+m \) is even under the following values of the parameters \( \phi_0 \) and \( w \).

i. \( \phi_0 = 0, \omega = \pm \pi / 2, \)

ii. \( \phi_0 = \pi / 2, \omega = \pi l = 0, 1, 2, \ldots \ldots \)

4. Perturbation of the 1st order
The existence of the periodic orbits analytic relative to \( \mu \) with period \( S = s^* + \delta s \) and equation of motion as
\[ \frac{dx_i}{d\zeta} = \frac{s^* + \alpha}{x^*} x_i \]
\[ (17) \]
Where,
\[ \zeta = \frac{S}{x^* + \Delta s}, S = \frac{s^*}{s^* + \alpha} S \]

The solution of the (16) may be sought in the form of series integral power of \( \mu \).
\[ xk = xk^{(0)} + \sum_{i=1}^{\infty} xk^{(i)} \mu^i, (k = 1, \ldots \ldots , A) \]
\[ (18) \]
For \( c \) and \( \alpha \), we have
\[ c = c_0 + \sum_{i=1}^{\infty} c_i \mu^i, \alpha = \sum_{i=1}^{\infty} \alpha_i \mu^i \]
\[ (19) \]
Let us substitute (17),(18) in (16) for the determination of the function $x_i (j)$ ($\zeta$) we have system of linear differential equation.

$$\frac{dx^{(i)}}{d\zeta} = \sum_{j=1}^{4} p_{kj} x_j + x k^{(i)} h_i + \phi_k c_i + F_k^{(i)}$$  \hspace{1cm} (20)

$$pkj = \frac{\partial x_k^{(0)}}{\partial x_j}, x_k^{(0)} = x k(x(0), x_4^{(0)}, c_0)$$

$$\phi_k = \frac{\partial x_k}{\partial c_o}$$

$$F_k^{(i)} \left(\{x_1^{(0)}, x_4^{(i=1)}, c_0, \ldots, (i = 1)\}\right)$$

In particular for $i=1$, we have

$$F_1^{(1)}(0) = 2\varepsilon_0 r_1 q$$
$$F_2^{(1)}(1) = -2 \varepsilon_0 r_1 p$$
$$F_3^{(1)} = -16 p^3 + \frac{8 p}{r_2} + \frac{8 p r_1 (1 - r_3)}{r_2} + 2 \varepsilon_0 r_1 Q - 4 \varepsilon_0 (q p - p Q) P - 24 \varepsilon_0 r_1^2 P$$
$$F_4^{(1)} = 16 q^3 + \frac{8 q}{r_2} - \frac{8 q r_1 (1 + r_3)}{r_2^2} - 2 \varepsilon_0 r_1 P - 4 \varepsilon_0 (q p - p Q) q - 24 \varepsilon_0 r_1^2 q$$

$$\phi_1 = \phi_2 = 0, \phi_3 = -4 P_0, \phi_4 = -4 q_0$$

$$x_1^{(0)} = P_0 + 2 r q_0, x_2^{(0)} = Q_0 - 2 r P_0; x_3^{(0)} = 2 r Q_0 - 4 P_0 C_0$$

$$x_4^{(0)} = 2 r P_0 - 4 q_0 C_0$$  \hspace{1cm} (21)

All the linearly independent solution of the

$$\frac{dy_i}{d\zeta} = \sum_{k=1}^{4} P_{ik} y_k$$  \hspace{1cm} (23)

Are obtained from the expression (13)

Let us take particular solution of the system (19) with the help of the method of arbitrary constant.

Let \( x_j^{(i)} = \sum_{k=1}^{4} L_k y_{jk} \),

Where \( \{y_{jk}\} \) is a fundamental system if solution of the homogeneous system (23)

$$\sum_{k=1}^{4} y_{jk} \frac{\partial y_k}{\partial \zeta} = \phi_j + x_j^{(i)} h_i + F_j^{(i)}$$

$$\zeta$$

We multiplying the expression (25) by $\zeta_{ji}$, where \( \{\zeta_{ji}\} \) is a matrix of solution for conjugate system (23) we shall add them in $j$ from 1 to 4 we have

$$\sum_{k=1}^{4} A_{ik} L_k = f_i$$

Where \( A_{ik} = \sum_{j=1}^{4} \zeta_{ji} y_{ik} \)

$$f_i = \sum_{k=1}^{4} (x_k^{(0)} h_1 + \phi_k C_1 + F_k^{(0)}) \zeta_{ki}$$

The expression (23) is a system in variation of a cannonical system, so far determination of $\zeta_{ki}$

We have the following relations

$$\zeta_{1i} = -y_{3i}, \zeta_{2i} = -y_{4i}, \zeta_{3i} = -y_{1i}, \zeta_{4i} = y_{2i}$$

As the coefficient $A_{ij}$ depends on time so far the determination we shall put $\zeta = 0$, using the formula (14) for the calculation of $y_{ij(0)}$, we find

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Thus we have the following system

\[ L_1 = \frac{\sqrt{C}}{4} f_1, L_3 = \frac{1}{8} f_2 - \frac{1}{2c_0} f_1 \]

\[ L_2 = \frac{1}{8} f_3; L_4 = \frac{-\sqrt{C}}{4} f_4 + \frac{1}{2c_0} f_3 \]

Integrating we get

\[ L_1 = \left( -\frac{\zeta \sigma^2}{4c_0} + \frac{3t_0}{4c_0} \right) C_1 - \frac{\sqrt{C}}{4} b_4 + \frac{1}{2c_0} b_3 \]

\[ L_2 = \zeta b_1 + \frac{t_0}{8} \zeta - \frac{1}{8} b_3 \]

\[ L_3 = -\left( \frac{-\sigma^2}{4} + \frac{\sigma^2(0)}{4} \right) C_1 + \frac{1}{8} b_2 - \frac{1}{2c_0} b_1 \]

\[ L_4 = \frac{\sqrt{C}}{4} b_1 \]

\[ b_1 = 16 \int_0^\zeta \left( pqr_1 \left( \frac{1}{r_2^3} - 1 \right) \right) d\zeta + 4\epsilon_0 \int_0^\zeta r_1 ppPd\zeta \]

\[ b_2 = \left( \frac{d\Omega}{d\mu} \right) \mu = 0 + \left( \frac{d\Omega}{d\mu} \right) \mu = 0, \zeta = 0 \]

\[ b_3 = \int_0^\zeta \left\{ -8(p^4 - q^4) \left[ 2 + \frac{1}{r_1^3} \right] + \frac{8r_1}{r_2^3} \right\} d\zeta + 16 \int_0^\zeta t p q r_1 \left( \frac{1}{r_2^3} - 1 \right) dt \]

\[ + \epsilon_0 \left\{ \int_0^\zeta r_1 (p Q - q p) d\zeta - 3 \int_0^\zeta r_2^3 d\zeta \right\} \]

\[ b_4 = \int_0^\zeta \left\{ \tan(2\sqrt{c} \zeta - \phi_0) - t/n \right\} \frac{db_1}{d\zeta} - \frac{2}{n} \frac{db_2}{d\zeta} d\zeta - \frac{16}{n} \epsilon_0 \left( \frac{c}{2} + \frac{1}{4\sqrt{c}} \sin 2\sqrt{c} \cos(2\sqrt{c} \zeta - 2\phi_0) \right) \]

(29)

In general solution of the system (19)

\[ x_k^{(1)} = \sum_{i=1}^4 (L_i + a_i)y_{ki}; \quad (k = 1, \ldots, 4) \]

(30)

\[ L_3 = -\left( \frac{-\sigma^2}{4} + \frac{\sigma^2(0)}{4} \right) C_1 + \frac{1}{8} b_2 - \frac{1}{2c_0} b_1 \]

\[ L_4 = \frac{\sqrt{C}}{4} b_1 \]

\[ b_1 = 16 \int_0^\zeta \left( pqr_1 \left( \frac{1}{r_2^3} - 1 \right) \right) d\zeta + 4\epsilon_0 \int_0^\zeta r_1 ppPd\zeta \]

\[ b_2 = \left( \frac{d\Omega}{d\mu} \right) \mu = 0 + \left( \frac{d\Omega}{d\mu} \right) \mu = 0, \zeta = 0 \]
\[ b_3 = \int_0^\zeta \left\{ -8(p^4 - q^4) \left( 2 + \frac{1}{r_1^2} \right) + \frac{8r_1}{r_2^2} \right\} \, d\zeta \]

\[ + 16 \int_0^\zeta t p q r_1 \left( \frac{1}{r_2^3} - 1 \right) \, dt + \varepsilon_0 \left\{ \int_0^\zeta r_1 (pQ - qp) \, d\zeta - 3 \int_0^\zeta r_2^3 \, d\zeta \right\} \]

\[ b_4 = \int_0^\zeta \left\{ \left( \tan(2\sqrt{c} \zeta - \phi_0) - \frac{t}{r} \right) \frac{db_1}{d\zeta} - \frac{2}{n} \zeta \frac{db_2}{d\zeta} \right\} \, d\zeta - \frac{16}{n} \varepsilon_0 \left\{ \frac{\zeta}{2} \right\} + \frac{1}{4\sqrt{c}} \sin(2\sqrt{c}\zeta) \cos(2\sqrt{c}\zeta) \]

\[ - 2\phi_0 \right\} \]

In general solution of the (19)

\[ x_k^{(1)} = \sum_{i=1}^4 (L_i + a_i) y_k_i; \quad (k=1, \ldots, 4) \]

there will enter six constant \( h_i, c, u_i \) \((i=1, \ldots, 4)\) which must be chosen in such a manner that the following relations are satisfied:

\[ x_2(0) = \frac{d\Omega}{d\mu} = x_3^0 x_4^0 (s^*), \mu = 0 \]

\[ X_2(0) = X_3(0) = \frac{d\Omega}{d\mu} = 0, \quad X_2^0 (s^*) = X_3^0 (s^*) = 0, \quad S^* = \frac{\pi K}{4\sqrt{c}} \]

If \( k+m \) is even

From the relation

\[ x_2(0) = X_3(0) = 0 \]

We find immediately that \( a_1 = a_2 = 0 \) we shall write the equation (30) and (31) in the following manner

\[ x_i^0 (s^*) = F_{i1} h_1 + F_{i2} c_1 + F_{i3} a_3 + F_{i4} a_4 + F_i^* = 0 \]

\[ X_j^0 (s^*) = F_{j1} h_1 + F_{j2} c_1 + F_{j3} a_3 + F_{j4} a_4 + F_j^* = 0 \]

(33)

Where

\[ i=1, j=4 \] if \( k+m \) is odd

\[ i=2, j=3 \] if \( k+m \) is even

\[ F_{K1} = C_{K2}, F_{K3} = Y_{K3}, F_{K4} = Y_{K4} \]

\[ F_K = \left( -\frac{\sqrt{c}}{4} b_4 + \frac{1}{2c_0} b_3 \right) y_{k1} - \frac{1}{8} b_3 y_{k2} + \left( \frac{1}{8} b_2 - \frac{1}{2} c_0 b_1 \right) y_{k3} + \frac{\sqrt{c_0}}{4} b_1, y_{k4} \]

(34)

Where \( hi, c_1, a_3, a_4 \) are defined by the formula (28)

Thus we have three equations for determinations of four constant \( (hi, c_1, a_3, a_4) \) one of them chosen arbitrarily and \( \emptyset_0 = 0, = \pm \pi / 2 \). One may required in order that in perturbed as well as in unperturbed motion the period is independent variable \( t \) may coincide.

**REFERENCES**